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ON THE CONSTRUCTION OF THREE-VALUED LUKASIEWICZ–MOISIL ALGEBRAS

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The author constructs a three-valued Lukasiewicz–Moisil algebra from a monadic three-valued Lukasiewicz–Moisil algebra, generalizing A. Monteiro's (1974) construction of a three-valued Lukasiewicz–Moisil algebra from a monadic Boolean algebra, and constructs a monadic three-valued Lukasiewicz–Moisil algebra from a monadic n-valued one, generalizing V. Boicescu's (1971) construction of a three-valued Lukasiewicz–Moisil algebra from an n-valued one, $n \geqslant 3$. Thus one can construct a three-valued Lukasiewicz–Moisil algebra from a monadic n-valued Lukasiewicz–Moisil algebra, $n \geqslant 2$.

Introduction

The notion of a three-valued Lukasiewicz algebra has been introduced by Moisil [8–10]. He has also introduced the notion of an n-valued Lukasiewicz algebra [9], called therefore, by Cignoli [12], a Moisil algebra of order n.

A. Monteiro [1] has found a construction $\mathscr{L}$ of a three-valued Lukasiewicz–Moisil algebra from a monadic Boolean algebra, using the theory of N-lattices (Nelson algebras). L. Monteiro and Coppola [5] give a direct proof of the result obtained in [1]. A. Monteiro [1] and L. Monteiro [13] have also proved that, given a three-valued Lukasiewicz–Moisil algebra L, there exists a monadic Boolean algebra A such that $\mathscr{L}(A)$, the three-valued Lukasiewicz–Moisil algebra obtained from A by the construction $\mathscr{L}$, is isomorphic with L.

Boicescu [3, 4] gives some constructions of a three-valued Lukasiewicz–Moisil algebra from an n-valued Lukasiewicz–Moisil algebra.

In Section 1 we give a construction $\mathscr{L}'$ of a three-valued Lukasiewicz–Moisil algebra from a monadic three-valued Lukasiewicz–Moisil algebra, generalizing the construction from [5]. In Section 2 we give some constructions of a monadic three-valued Lukasiewicz–Moisil algebra from a monadic n-valued Lukasiewicz–Moisil algebra, extending the results from [3, 4] to monadic algebras.

1. A construction of a three-valued Lukasiewicz–Moisil algebra from a monadic three-valued Lukasiewicz–Moisil algebra

Definitions and properties

We assume the reader to be familiar with the definitions and the properties of a distributive lattice and of a Boolean algebra.

Definition. A *three-valued Lukasiewicz–Moisil algebra* (3-L.M.A.) is an algebra $(A; \vee, \wedge, -, s_2, 1)$, where A is a non-empty set, $\vee$ and $\wedge$ are binary operations, $-$ and s_2 are unary operations, $1 \in A$ (nullary operation), satisfying the following axioms [2]:

(S1) $x \wedge (x \vee y) = x$,

(S2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,

(N1) $--x = x$,

(N2) $-(x \wedge y) = -x \vee -y$,

(L0) $x \vee 1 = 1$,

(L1) $-x \vee s_2 x = 1$,

(L2) $x \wedge -x = -x \wedge s_2 x$,

(L3) $s_2(x \wedge y) = s_2 x \wedge s_2 y$.

L. Monteiro [6] has proved that axiom (L0) is dependent on the others and that the axioms (S1), (S2), (N1), (N2), (L1), (L2) and (L3) are independent.

From (S1) and (S2) (cf. [11]) we have that A is a distributive lattice, with last element 1 (from (L0). If we put $0 = -1$, then 0 is the first element. From (N1) and (N2) we have that the operation $-$ is an involutive duality (see (L4)–(L15) below).

Let us denote by s_1 the operation defined on A, for every x, by

$$s_1 x = -s_2(-x). \tag{1}$$

Definition. A *monadic three-valued Lukasiewicz–Moisil algebra* (monadic 3-L.M.a.) is a pair $(A, \exists)$, where A is a 3-L.M.a. and $\exists : A \to A$ is a mapping, called an existential quantifier, such that, for every $x, y \in A$, the following axioms hold [7]:

(E0) $\exists 0 = 0$,

(E1) $x \wedge \exists x = x$, or equivalent $x \vee \exists x = \exists x$, or $x \leqslant \exists x$,

(E2) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

(E3) $s_1 \exists x = \exists s_1 x$ and $s_2 \exists x = \exists s_2 x$.

Let $\forall : A \to A$ be the mapping, called a universal quantifier, defined by

$$\forall x = -\exists(-x). \tag{2}$$

We now give some properties [12] verified in a 3-L.M.a.:

(L4) $x \wedge 0 = 0, \quad x \wedge 1 = x, \quad x \vee 0 = x,$

(L5) $s_i(x \vee y) = s_i x \vee s_i y, \quad i = 1, 2,$

(L6) $s_i(x \wedge y) = s_i x \wedge s_i y, \quad i = 1, 2,$

(L7) $s_i x \vee -s_i x = 1, \quad i = 1, 2,$

(L8) $s_i x \wedge -s_i x = 0, \quad i = 1, 2,$

(L9) $s_i s_j x = s_j x, \quad i, j = 1, 2,$

(L10) $s_1(-x) = -s_2 x, \quad s_2(-x) = -s_1 x,$

(L11) $s_1 x \leqslant x \leqslant s_2 x,$

(L12) $-(x \vee y) = -x \wedge -y,$

(L13) $s_i 0 = 0, \quad s_i 1 = 1, \quad i = 1, 2,$

(L14) $-x \vee s_2 x = 1, \quad x \wedge s_1(-x) = 0,$

(L15) The set $B(A) = \{x \in A \mid x = s_i x, i = 1, 2\}$ is a Boolean algebra, with the induced operations $\vee, \wedge, -, 0$ and 1, and in a Boolean algebra we have $x \leqslant y \Leftrightarrow -x \vee y = 1 \Leftrightarrow x \wedge -y = 0$.

In a monadic 3-L.M.a. there are more properties [7], e.g.,

(U0) $\forall 1 = 1,$

(U1) $\forall x \leqslant x,$

(U2) $\forall(x \vee \forall y) = \forall x \vee \forall y,$

(U3) $s_1 \forall x = \forall s_1 x$ and $s_2 \forall x = \forall s_2 x,$

(E4) $-\exists x = \forall(-x),$	(U4) $-\forall x = \exists(-x),$
(E5) $\exists 1 = 1,$	(U5) $\forall 0 = 0,$
(E6) $\exists\exists x = \exists x,$	(U6) $\forall\forall x = \forall x,$
(E7) If $x \leqslant y$, then $\exists x \leqslant \exists y,$	(U7) If $x \leqslant y$, then $\forall x \leqslant \forall y,$
(E8) $y \leqslant \exists x \Leftrightarrow \exists y \leqslant \exists x,$	(U8) $\forall x \leqslant y \Leftrightarrow \forall x \leqslant \forall y,$
(E9) $-x \vee s_2 \exists x = 1,$	(U9) $-x \wedge s_1 \forall x = 0,$
(E10) $\exists x \vee s_2(-x) = 1,$	(U10) $\forall x \wedge s_1(-x) = 0,$
(E11) $\exists(x \vee y) = \exists x \vee \exists y,$	(U11) $\forall(x \wedge y) = \forall x \wedge \forall y,$
(E12) $\exists(x \wedge \forall y) = \exists x \wedge \forall y,$	(U12) $\forall(x \vee \exists y) = \forall x \vee \exists y,$
(E13) $\exists \forall x = \forall x,$	(U13) $\forall \exists x = \exists x,$

(E14) $\forall s_1 x \leqslant s_1 x \leqslant x \leqslant s_2 x \leqslant \exists s_2 x,$

(E15) $-s_i \exists x = -\exists s_i x = \forall s_j(-x)$, and
$-\forall s_i x = \exists s_j(-x) = s_j \exists(-x), \quad i, j = 1, 2, i \neq j.$

Construction $\mathscr{L}'$

Let $(A, \exists)$ be a monadic 3-L.M.a. and $\rightarrow\!\!\!\!\!\ominus\!\!\!\!\rightarrow$ be a binary operation defined on A by

$$x \,{-}\!\!\ominus\!\!\!\rightarrow y = \exists s_2(-x) \vee y. \tag{3}$$

Let us define on A a binary operation $\rightarrow$ by

$$x \rightarrow y = (x \mathrel{-\!\ominus\!\!\rightarrow} y) \wedge (-y \mathrel{-\!\ominus\!\!\rightarrow} -x). \tag{4}$$

We put, by definition,

$$x \cup y = (x \rightarrow y) \rightarrow y, \tag{5}$$

$$x \cap y = -(-x \cup -y). \tag{6}$$

Lemma 1. *For every $x, y \in A$ we have*

$$x \cup y = \forall s_1 x \vee y \vee (x \wedge \forall s_1(-y)), \tag{7}$$

$$x \cup y = (x \vee y) \wedge (\forall s_1 x \vee y \vee \forall s_1(-y)), \tag{8}$$

$$x \cap y = \exists s_2 x \wedge y \wedge (x \vee \exists s_2(-y)), \tag{9}$$

$$x \cap y = (x \wedge y) \vee (\exists s_2 x \wedge y \wedge \exists s_2(-y)). \tag{10}$$

Proof. By distributivity, (E14) and absorption, we obtain

$$x \rightarrow y = (\exists s_2(-x) \wedge \exists s_2 y) \vee y \vee -x. \tag{4'}$$

If we put $x \rightarrow y = D$, then from (5) and (4′) we obtain

$$x \cup y = D \rightarrow y = (\exists s_2(-D) \wedge \exists s_2 y) \vee y \vee -D;$$

putting $B = \exists s_2(-D) \wedge \exists s_2 y$ and $C = y \vee -D$, we have $x \cup y = B \vee C$.

But, using (E15), distributivity, (L5), (L6), (U3), (L9) and (E14), we obtain

$$\begin{aligned} B &= \exists s_2(x \wedge -y \wedge (\forall s_1 x \vee \forall s_1(-y))) \wedge \exists s_2 y \\ &= \exists((s_2 x \wedge s_2(-y) \wedge \forall s_1 x) \vee (s_2 x \wedge s_2(-y) \wedge \forall s_1(-y))) \wedge \exists s_2 y \\ &= \exists((\forall s_1 x \wedge s_2(-y)) \vee (s_2 x \wedge \forall s_1(-y))) \wedge \exists s_2 y. \end{aligned}$$

By (E11) and (E12), by distributivity, (E15), (E3), (L8) and (L4), we obtain

$$\begin{aligned} B &= ((\exists s_2(-y) \wedge \forall s_1 x) \vee (\exists s_2 x \wedge \forall s_1(-y))) \wedge \exists s_2 y \\ &= (\forall s_1 x \wedge \exists s_2(-y) \wedge \exists s_2 y) \vee (\exists s_2 x \wedge \forall s_1(-y) \wedge \exists s_2 y) \\ &= (\forall s_1 x \wedge \exists s_2(-y) \wedge \exists s_2 y) \vee (\exists s_2 x \wedge -\exists s_2 y \wedge \exists s_2 y) \\ &= \forall s_1 x \wedge \exists s_2(-y) \wedge \exists s_2 y. \end{aligned}$$

On the other hand, by (E15), distributivity and (E14), we have that

$$\begin{aligned} C = -D \vee y &= (x \wedge -y \wedge (\forall s_1 x \vee \forall s_1(-y))) \vee y \\ &= (x \wedge -y \wedge \forall s_1 x) \vee (x \wedge -y \wedge \forall s_1(-y)) \vee y \\ &= (\forall s_1 x \wedge -y) \vee (x \wedge \forall s_1(-y)) \vee y. \end{aligned}$$

Hence,

$$
\begin{aligned}
x \cup y &= B \vee C \\
&= (\forall s_1 x \wedge \exists s_2(-y) \wedge \exists s_2 y) \vee (\forall s_1 x \wedge -y) \vee (x \wedge \forall s_1(-y)) \vee y \\
&= (\forall s_1 x \wedge (-y \vee (\exists s_2(-y) \wedge \exists s_2 y))) \vee (x \wedge \forall s_1(-y)) \vee y \\
&= (\forall s_1 x \wedge (-y \vee \exists s_2(-y)) \wedge (-y \vee \exists s_2 y)) \vee (x \wedge \forall s_1(-y)) \vee y,
\end{aligned}
$$

using distributivity.

Further on, by (E14), (E3) and (E9), we have that

$$x \cup y = (\forall s_1 x \wedge \exists s_2(-y)) \vee (x \wedge \forall s_1(-y)) \vee y.$$

Therefrom we obtain, using distributivity, then (E3) and (E9),

$$
\begin{aligned}
x \cup y &= ((\forall s_1 x \vee y) \wedge (\exists s_2(-y) \vee y)) \vee (x \wedge \forall s_1(-y)) \\
&= \forall s_1 x \vee y \vee (x \wedge \forall s_1(-y)),
\end{aligned}
$$

hence (7).

Therefrom, using distributivity and (E14), we have

$$x \cup y = (x \vee y) \wedge (\forall s_1 x \vee y \vee \forall s_1(-y)),$$

hence (8).

We obtain (9) from (6), using successively (L12), (N2), (N1) and (E15), and (10) is obtained from (9), using distributivity and (E14). □

We define a binary relation $\equiv$ on A by

$$
x \equiv y \Leftrightarrow \begin{cases}
s_1 x \mathrel{-\!\!\ominus\!\!\rightarrow} s_1 y = 1, \\
s_1 y \mathrel{-\!\!\ominus\!\!\rightarrow} s_1 x = 1, \\
-s_2 y \mathrel{-\!\!\ominus\!\!\rightarrow} -s_2 x = 1, \\
-s_2 x \mathrel{-\!\!\ominus\!\!\rightarrow} -s_2 y = 1.
\end{cases}
\tag{11}
$$

The right-hand side is equivalent to

$$
\begin{cases}
\exists s_2(-s_1 x) \vee s_1 y = 1 \\
\exists s_2(-s_1 y) \vee s_1 x = 1 \\
\exists s_2 s_2 y \vee -s_2 x = 1 \\
\exists s_2 s_2 x \vee -s_2 y = 1
\end{cases}
\Leftrightarrow
\begin{cases}
\exists s_2(-x) \vee s_1 y = 1 \\
\exists s_2(-y) \vee s_1 x = 1 \\
\exists s_2 y \vee -s_2 x = 1 \\
\exists s_2 x \vee -s_2 y = 1
\end{cases}
\Leftrightarrow
\begin{cases}
-\forall s_1 x \vee s_1 y = 1 \\
-\forall s_1 y \vee s_1 x = 1 \\
\exists s_2 y \vee -s_2 x = 1 \\
\exists s_2 x \vee -s_2 y = 1
\end{cases}
$$

$$
\Leftrightarrow
\begin{cases}
\forall s_1 x \leqslant s_1 y \\
\forall s_1 y \leqslant s_1 x \\
s_2 x \leqslant \exists s_2 y \\
s_2 y \leqslant \exists s_2 x
\end{cases}
\Leftrightarrow
\begin{cases}
\forall s_1 x \leqslant \forall s_1 y \\
\forall s_1 y \leqslant \forall s_1 x \\
\exists s_2 x \leqslant \exists s_2 y \\
\exists s_2 y \leqslant \exists s_2 x
\end{cases}
\Leftrightarrow
\begin{cases}
\forall s_1 x = \forall s_1 y \\
\exists s_2 x = \exists s_2 y
\end{cases}
$$

using successively (3), (L10) and (L9), (E15), (L15), (U8) and (E8). Consequently,

$$x \equiv y \quad \text{if and only if} \quad \exists s_2 x = \exists s_2 y \ \text{ and } \ \forall s_1 x = \forall s_1 y. \tag{12}$$

Theorem 1. *The relation* $\equiv$, *defined on* A, *is an equivalence relation, which satisfies the substitution property for the operations* $\cup, \cap, -, \exists s_2, \forall s_1$.

Let $L = A/_{\equiv}$ *be the quotient set and* $\hat{x}$ *be the equivalence class containing* $x \in A$. *If we put* $\hat{x} \cup \hat{y} = (x \cup y)^\wedge$, $\hat{x} \cap \hat{y}) = (x \cap y)^\wedge$, $\sim\hat{x} = (-x)^\wedge$, $\sigma_2\hat{x} = (\exists s_2 x)^\wedge$, $\sigma_1\hat{x} = (\forall s_1 x)^\wedge$ *and* $I = \hat{1}$, *then the system* $(L; \cup, \cap, \sim, \sigma_2, I)$ *is a* 3-*L.M.a.*

Proof. The proof being long, we decompose it into a few lemmas.

Lemma 2. *For every* $x, y \in A$ *we have*

$$-(x \cup y) = -x \cap -y \quad \textit{and} \quad -(x \cap y) = -x \cup -y.$$

Proof. Using (7), (E15) and (9), we obtain

$$\begin{aligned} -(x \cup y) &= -\forall s_1 x \wedge (-x \vee -\forall s_1(-y)) \wedge -y \\ &= \exists s_2(-x) \wedge (-x \vee \exists s_2 y) \wedge -y = -x \cap -y. \end{aligned}$$

Using (9), (E15) and (7), we obtain, similarly, $-(x \cap y) = -x \cup -y$. □

Lemma 3. $\exists s_2(x \cup y) = \exists s_2 x \vee \exists s_2 y$ *for every* $x, y \in A$.

Proof. Applying successively (7), then (L5) and (E11), then (L6), (U3), then (L9), (E13), (E12), then (E15), then (E3), distributivity and (L15) and finally (E14), we obtain

$$\begin{aligned} \exists s_2(x \cup y) &= \exists s_2(\forall s_1 x \vee (x \wedge \forall s_1(-y)) \vee y) \\ &= \exists s_2 \forall s_1 x \vee \exists s_2(x \wedge \forall s_1(-y)) \vee \exists s_2 y \\ &= \exists \forall s_2 s_1 x \vee \exists(s_2 x \wedge \forall s_2 s_1(-y)) \vee \exists s_2 y \\ &= \forall s_1 x \vee (\exists s_2 x \wedge \forall s_1(-y)) \vee \exists s_2 y \\ &= \forall s_1 x \vee (\exists s_2 x \wedge -\exists s_2 y) \vee \exists s_2 y \\ &= \forall s_1 x \vee (\exists s_2 x \vee \exists s_2 y) = \exists s_2 x \vee \exists s_2 y. \quad \square \end{aligned}$$

Lemma 4. $\forall s_1(x \cap y) = \forall s_1 x \wedge \forall s_1 y$ *for every* $x, y \in A$.

Proof. Using (E15), Lemma 2, Lemma 3, (L12) and (E15), we obtain

$$\begin{aligned} \forall s_1(x \cap y) &= -\exists s_2(-(x \cap y)) = -\exists s_2(-x \cup -y) \\ &= -(\exists s_2(-x) \vee \exists s_2(-y)) = -\exists s_2(-x) \wedge -\exists s_2(-y) \\ &= \forall s_1 x \wedge \forall s_1 y. \quad \square \end{aligned}$$

Lemma 5. $\exists s_2(x \cap y) = \exists s_2 x \wedge \exists s_2 y$ *for every* $x, y \in A$.

Proof.

$$
\begin{aligned}
&\exists(x \cap y) = \\
&\quad = \exists((x \wedge y) \vee (\exists s_2 x \wedge y \wedge \exists s_2(-y))) \quad \text{by (10)} \\
&\quad = \exists(x \wedge y) \vee \exists(\exists s_2 x \wedge y \wedge \exists s_2(-y)) \quad \text{by (E11)} \\
&\quad = \exists(x \wedge y) \vee (\exists(y \wedge \exists s_2 x) \wedge \exists s_2(-y)) \quad \text{by (E2)} \\
&\quad = \exists(x \wedge y) \vee (\exists y \wedge \exists s_2 x \wedge \exists s_2(-y)) \quad \text{by (E2)} \\
&\quad = (\exists(x \wedge y) \vee \exists y) \wedge (\exists(x \wedge y) \vee \exists s_2 x) \wedge (\exists(x \wedge y) \vee \exists s_2(-y)) \\
&\qquad \text{using distributivity} \\
&\quad = \exists((x \wedge y) \vee y) \wedge \exists((x \wedge y) \vee s_2 x) \wedge \exists((x \wedge y) \vee s_2(-y)) \quad \text{by (E11)} \\
&\quad = \exists y \wedge \exists((x \vee s_2 x) \wedge (y \vee s_2 x)) \wedge \exists(x \vee s_2(-y)) \wedge (y \vee s_2(-y))) \\
&\qquad \text{by idempotency and distributivity} \\
&\quad = \exists y \wedge \exists(s_2 x \wedge (y \vee s_2 x)) \wedge \exists(x \vee s_2(-y)) \quad \text{by (E14), (L14), (L4)} \\
&\quad = \exists y \wedge \exists s_2 x \wedge \exists(x \vee s_2(-y)) \quad \text{by idempotency} \\
&\quad = \exists y \wedge \exists s_2 x \wedge (\exists x \vee \exists s_2(-y)) \quad \text{by (E11)} \\
&\quad = (\exists y \wedge \exists s_2 x \wedge \exists x) \vee (\exists y \wedge \exists s_2 x \wedge \exists s_2(-y)) \quad \text{using distributivity} \\
&\quad = (\exists x \wedge \exists y) \vee (\exists s_2 x \wedge \exists y \wedge \exists s_2(-y)) \quad \text{by (E3), (E14).}
\end{aligned}
$$

Using successively (E3), the relation obtained, then (L5) and (L6), then (E3) and idempotency, we have

$$
\begin{aligned}
\exists s_2(x \cap y) &= s_2 \exists(x \cap y) \\
&= s_2((\exists x \wedge \exists y) \vee (\exists s_2 x \wedge \exists y \wedge \exists s_2(-y))) \\
&= (s_2 \exists x \wedge s_2 \exists y) \vee (\exists s_2 x \wedge s_2 \exists y \wedge \exists s_2(-y)) = \exists s_2 x \wedge \exists s_2 y. \quad \square
\end{aligned}
$$

Lemma 6. $\forall s_1(x \cup y) = \forall s_1 x \vee \forall s_1 y$, *for every* $x, y \in A$.

Proof.

$$
\begin{aligned}
\forall s_1(x \cup y) &= -\exists s_2(-(x \cup y)) = -\exists s_2(-x \cap -y) \\
&= -(\exists s_2(-x) \wedge \exists s_2(-y)) = -\exists s_2(-x) \vee -\exists s_2(-y) \\
&= \forall s_1 x \vee \forall s_1 y,
\end{aligned}
$$

using successively (E15), Lemma 2, Lemma 5, (N2) and (E15). $\square$

Lemma 7. *The relation* $\equiv$, *defined on* A *by* (11), *is an equivalence relation.*

Lemma 8. *The equivalence relation* $\equiv$ *satisfies the substitution property for the operations* $\cup, \cap, -, \exists s_2, \forall s_1$.

Proof. Let $x, x', y, y' \in A$, with $x \equiv x'$ and $y \equiv y'$. Then, using (12) and Lemmas 3 and 6, we obtain

$$\exists s_2(x \cup y) = \exists s_2 x \vee \exists s_2 y = \exists s_2 x' \vee \exists s_2 y' = \exists s_2(x' \cup y')$$

and

$$\forall s_1(x \cup y) = \forall s_1 x \vee \forall s_1 y = \forall s_1 x' \vee \forall s_1 y' = \forall s_1(x' \cup y'),$$

which is the substitution property for $\cup$, and similarly, using (12) and Lemmas 5 and 4, we obtain

$$\exists s_2(x \cap y) = \exists s_2 x \wedge \exists s_2 y = \exists s_2 x' \wedge \exists s_2 y' = \exists s_2(x' \cap y')$$

and

$$\forall s_1(x \cap y) = \forall s_1 x \wedge \forall s_1 y = \forall s_1 x' \wedge \forall s_1 y' = \forall s_1(x' \cap y'),$$

which is the substitution property for $\cap$.

By (12) and (E15), we obtain

$$\exists s_2(-x) = -\forall s_1 x = -\forall s_1 x' = \exists s_2(-x')$$

and

$$\forall s_1(-x) = -\exists s_2 x = -\exists s_2 x' = \forall s_1(-x'),$$

which is the substitution property for $-$.

By assumption and (12), we have, directly,

$$\exists s_2(\exists s_2 x) = \exists s_2(\exists s_2 x') \quad \text{and} \quad \forall s_1(\exists s_2 x) = \forall s_1(\exists s_2 x'),$$

$$\exists s_2(\forall s_1 x) = \exists s_2(\forall s_1 x') \quad \text{and} \quad \forall s_1(\forall s_1 x) = \forall s_1(\forall s_1 x'),$$

which is the substitution property for $\exists s_2, \forall s_1$. $\square$

Remark. The equivalence class of 1 contains only the element 1, as results using (E14).

Lemma 9. *The system* $(L = A/_{\equiv}; \cup, \cap)$ *is a distributive lattice.*

Proof. It is sufficient [11] to verify the properties

(S′1) $\hat{x} \cap (\hat{x} \cup \hat{y}) = \hat{x}$,

(S′2) $\hat{x} \cap (\hat{y} \cup \hat{z}) = (\hat{z} \cap \hat{x}) \cup (\hat{y} \cap \hat{x})$.

By definition of $\cap, \cup$ on L and by (12) and using Lemmas 5 and 3, and Lemmas 4 and 6, respectively with (S1), we obtain

$$\exists s_2(x \cap (x \cup y)) = \exists s_2 x \wedge (\exists s_2 x \vee \exists s_2 y) = \exists s_2 x$$

and

$$\forall s_1(x \cap (x \cup y)) = \forall s_1 x \wedge (\forall s_1 x \vee \forall s_1 y) = \forall s_1 x,$$

hence (S′1).

Similarly, using Lemmas 5 and 3, and Lemmas 4 and 6, respectively with (S2), we obtain

$$\begin{aligned}\exists s_2(x \cap (y \cup z)) &= \exists s_2 x \wedge (\exists s_2 y \vee \exists s_2 z)\\ &= (\exists s_2 z \wedge \exists s_2 x) \vee (\exists s_2 y \wedge \exists s_2 x)\\ &= \exists s_2((z \cap x) \cup (y \cap x))\end{aligned}$$

and

$$\begin{aligned}\forall s_1(x \cap (y \cup z)) &= \forall s_1 x \wedge (\forall s_1 y \vee \forall s_1 z)\\ &= (\forall s_1 z \wedge \forall s_1 x) \vee (\forall s_1 y \wedge \forall s_1 x)\\ &= \forall s_1((z \cap x) \cup (y \cap x)),\end{aligned}$$

hence (S′2). □

Lemma 10. *In the lattice* $(L; \cup, \cap)$, $I = \hat{1}$ *is the last element, i.e., it verifies*

(L′0) $\hat{x} \cup \hat{1} = \hat{1}$ *for every* $\hat{x} \in L$.

Proof. Using Lemma 3, (L13), (E5), and Lemma 6, (L13), (U0) respectively with (L0), we obtain

$$\exists s_2(x \cup 1) = \exists s_2 x \vee \exists s_2 1 = \exists s_2 x \vee 1 = 1 = \exists s_2 1$$

and

$$\forall s_1(x \cup 1) = \forall s_1 x \vee \forall s_1 1 = \forall s_1 x \vee 1 = 1 = \forall s_1 1,$$

i.e., (L′0). □

Lemma 11. *In the lattice* $(L; \cup, \cap)$, *the operation* $\sim$, *defined on* L, *verifies*

(N′1) $\sim\sim\hat{x} = \hat{x}$,

(N′2) $\sim(\hat{x} \cap \hat{y}) = \sim\hat{x} \cup \sim\hat{y}$.

Proof. The proof immediately follows from the definitions, (N1) and Lemma 2. □

Lemma 12. *In the lattice* $(L; \cup, \cap)$, *the operations* $\sim, \sigma_2$ *and* $I = \hat{1}$ *verify*

(L′1) $\sim\hat{x} \cup \sigma_2\hat{x} = \hat{1}$,

(L′2) $\hat{x} \cap \sim\hat{x} = \sim\hat{x} \cap \sigma_2\hat{x}$,

(L′3) $\sigma_2(\hat{x} \cap \hat{y}) = \sigma_2\hat{x} \cap \sigma_2\hat{y}$.

Proof. Using Lemma 3, then (E15), (E3), then (E6), (L9), then (U3), (E3), (L11), (L15) and finally (L13), (E5), we obtain

$$\begin{aligned}\exists s_2(-x \cup \exists s_2 x) &= \exists s_2(-x) \vee \exists s_2(\exists s_2 x) = -\forall s_1 x \vee \exists\exists s_2 s_2 x\\ &= -\forall s_1 x \vee \exists s_2 x = 1 = \exists s_2 1.\end{aligned}$$

Similarly, using Lemma 6, then (E15), (E3), then (E3), (U13), (L9), then (E3), (L7) and finally (L13), (U0), we obtain

$$\begin{aligned}\forall s_1(-x \cup \exists s_2 x) &= \forall s_1(-x) \vee \forall s_1(\exists s_2 x) = -\exists s_2 x \vee \forall\exists s_1 s_2 x\\ &= -s_2\exists x \vee \exists s_2 x = 1 = \forall s_1 1,\end{aligned}$$

hence (L′1).

To prove (L′2), first of all, using Lemma 5, then (E3), then (E6), (L9), then Lemma 5, we obtain

$$\begin{aligned}\exists s_2(-x \cap \exists s_2 x) &= \exists s_2(-x) \wedge \exists s_2 \exists s_2 x = \exists s_2(-x) \wedge \exists\exists s_2 s_2 x\\ &= \exists s_2(-x) \wedge \exists s_2 x = \exists s_2(x \cap -x).\end{aligned}$$

Secondly, using Lemma 4, then (E15), (E3), then (U13), (L9) and finally (E3), (L8), we obtain

$$\begin{aligned}\forall s_1(-x \cap \exists s_2 x) &= \forall s_1(-x) \wedge \forall s_1(\exists s_2 x) = -\exists s_2 x \wedge \forall\exists s_1 s_2 x\\ &= -\exists s_2 x \wedge \exists s_2 x = 0,\end{aligned}$$

and using Lemma 4, then (E15), then (U3), (E3), then (E14), (L15), we obtain

$$\forall s_1(x \cap -x) = \forall s_1 x \wedge \forall s_1(-x) = \forall s_1 x \wedge -\exists s_2 x = s_1 \forall x \wedge -s_2 \exists x = 0,$$

hence $\forall s_1(-x \cap \exists s_2 x) = \forall s_1(x \cap -x)$, hence (L′2).

To prove (L′3), we remark that, using (9), (E3), (E6), (L9), (S1) and Lemma 5, we have

$$\begin{aligned}\exists s_2 x \cap \exists s_2 y &= \exists s_2 \exists s_2 x \wedge (\exists s_2 x \vee \exists s_2(-\exists s_2 y)) \wedge \exists s_2 y\\ &= \exists\exists s_2 s_2 x \wedge (\exists s_2 x \vee \exists s_2(-\exists s_2 y)) \wedge \exists s_2 y\\ &= \exists s_2 x \wedge (\exists s_2 x \vee \exists s_2(-\exists s_2 y)) \wedge \exists s_2 y = \exists s_2 x \wedge \exists s_2 y\\ &= \exists s_2(x \cap y),\end{aligned}$$

hence (L′3). □

Proof of Theorem 1 (*continued*). By Lemmas 9, 10, 11 and 12 we obtain that the system $(L = A/_{\equiv};\ \cup, \cap, \sim, \sigma_2, I = \hat{1})$ is a 3-L.M.a. □

The next property is verified (see (1)),

$$\sigma_1 \hat{x} = \sim\sigma_2 \sim \hat{x}. \tag{1′}$$

Indeed, using the definition of σ_1, (E15), the definitions of $\sim$ and σ_2, we have that

$$\sigma_1 \hat{x} = (\forall s_1 x)^\wedge = (-\exists s_2(-x))^\wedge = \sim(\exists s_2(-x))^\wedge = \sim\sigma_2(-x)^\wedge = \sim\sigma_2 \sim \hat{x}.$$

Remarks. (1) If $s_1 x = s_2 x = \mathrm{Id}\, x = x$, hence, if we have a monadic Boolean algebra (i.e., a monadic 2-valued Lukasiewicz–Moisil algebra) instead of a monadic 3-L.M.a., the construction and the theorem we gave are reduced to those from [5].

(2) If $(A; \vee, \wedge, -, s_2, 1, \exists)$ is a monadic 3-L.M.a., then $(A; \cup, \cap, -, \exists s_2, 1)$ is not a 3-L.M.a. (for example, (L2) is not verified), where $\cup$, $\cap$ are those defined by (5), (6).

(3) If $(A; \vee, \wedge, -, s_2, 1, \exists)$ is a monadic 3-L.M.a., then $(B(A), \exists)$ is a monadic Boolean algebra (see (L15)). The construction from [1, 5] gives a 3-L.M.a. $L' = B(A)/_{\equiv}$. We can prove that L' is a subalgebra of the 3-L.M.a. $L = A/_{\equiv}$.

(4) Given a 3-L.M.a. L, there exists a monadic 3-L.M.a. A, such that a subalgebra of $\mathscr{L}'(A)$, the 3-L.M.a. obtained from A by the above $\mathscr{L}'$ construction, is isomorphic with L. The proof of this result is based on [13] and it will be published elsewhere.

2. Constructions of a monadic three-valued Lukasiewicz-Moisil algebra from a monadic n-valued Lukasiewicz-Moisil algebra

Construction 1

We assume the reader to be familiar with the definition of an n-valued Lukasiewicz–Moisil algebra (n-L.M.a.), $n \geqslant 2$ [12].

Definition. *A monadic n-valued Lukasiewicz–Moisil algebra* (monadic n-L.M.a.), $n \geqslant 2$, is a pair $(A', \exists)$, where A' is an n-L.M.a. $(A'; \cup, \cap, \neg, r_1, r_2, \ldots, r_{n-1}, \mathbf{1})$ and $\exists : A' \to A'$ is a mapping, called an *existential quantifier*, such that, for every $x, y \in A'$, the following axioms hold:

(F0) $\exists \mathbf{0} = \mathbf{0}$, *where* $\mathbf{0} = \neg \mathbf{1}$,

(F1) $x \leqslant \exists x$,

(F2) $\exists(x \cap \exists x) = \exists x \cap \exists y$,

(F3) $\exists r_i x = r_i \exists x$, $i = 1, \ldots, n-1$.

Let $(A', \exists)$ be a monadic n-L.M.a., $n \geqslant 3$. In the n-L.M.a. A' we put [3, 4]

$$
\begin{aligned}
x \to y &= r_{n-1} \neg x \cup y, \\
x \rightarrowtail y &= (x \to y) \cap (\neg y \to \neg x), \\
x \vee y &= (x \rightarrowtail y) \rightarrowtail y = r_1 x \cup y \cup (x \cap r_1 \neg y) \\
&= (x \cup y) \cap (r_1 x \cup y \cup r_1 \neg y), \\
x \wedge y &= \neg(\neg x \vee \neg y) = r_{n-1} x \cap y \cap (x \cup r_{n-1} \neg y) \\
&= (x \cap y) \cup (r_{n-1} x \cap r_{n-1} \neg y), \\
xRy &\Leftrightarrow (x \rightarrowtail y = \mathbf{1} \text{ and } y \rightarrowtail x = \mathbf{1}) \\
&\Leftrightarrow (r_1 x = r_1 y \text{ and } r_{n-1} x = r_{n-1} y).
\end{aligned}
$$

Theorem 2. *Let* $(A', \exists)$ *be a monadic n-L.M.a.,* $n \geqslant 3$. *The relation* R *is an equivalence relation on* A', *which satisfies the substitution property for* $\vee, \wedge, \neg, r_1, r_{n-1}, \exists$.

Let $A = A'/_R$ *be the quotient set. If we put* $\hat{x} \vee \hat{y} = (x \vee y)^\wedge$, $\hat{x} \wedge \hat{y} = (x \wedge y)^\wedge$, $-\hat{x} = (\neg x)^\wedge$, $s_1\hat{x} = (r_1 x)^\wedge$, $s_2\hat{x} = (r_{n-1}x)^\wedge$, $1 = \hat{\mathbf{1}} = \{\mathbf{1}\}$ *and* $\exists\hat{x} = (\exists x)^\wedge$, *then,*

(a) *the system* $(A : \vee, \wedge, -, s_2, 1)$ *is a* 3-*L.M.a,* *and*

(b) *the pair* $(A, \exists)$ *is a monadic* 3-*L.M.a.*

Proof. It is proved in [3, 4] that R is an equivalence relation, satisfying the substitution property for $\vee, \wedge, \neg, r_1, r_{n-1}$. Let xRx', $x, x' \in A'$. By definition of R and (F3), we obtain

$$r_1\exists x = \exists r_1 x = \exists r_1 x' = r_1 \exists x' \quad \text{and} \quad r_{n-1}\exists x = \exists r_{n-1}x = \exists r_{n-1}x' = r_{n-1}\exists x',$$

hence the substitution property for $\exists$.

The proof of (a) is done in [3, 4]. To prove (b), we have to verify (E0), (E1), (E2) and (E3) for every $\hat{x}, \hat{y} \in A$, where $\hat{\mathbf{0}} = -\hat{\mathbf{1}} = (\neg\mathbf{1})^\wedge$. So, $r_1\exists\mathbf{0} = r_1\mathbf{0}$ and $r_{n-1}\exists\mathbf{0} = r_{n-1}\mathbf{0}$, by (F0), hence (E0).

Further, we have

$$\begin{aligned} r_1(x \wedge \exists x) &= r_1((x \cap \exists x) \cup (r_{n-1}x \cap \exists x \cap r_{n-1}\neg\exists x)) \\ &= r_1(x \cup (r_{n-1}x \cap \exists x \cap r_{n-1}\neg\exists x)) \\ &= r_1 x \cup (r_{n-1}x \cap r_1\exists x \cap r_{n-1}\neg\exists x) \\ &= r_1 x \cup (r_{n-1}x \cap r_1\exists x \cap \neg r_1\exists x) \\ &= r_1 x \cup (r_{n-1}x \cap \mathbf{0}) = r_1 x \cup \mathbf{0} = r_1 x, \end{aligned}$$

and

$$\begin{aligned} r_{n-1}(x \wedge \exists x) &= r_{n-1}((x \cap \exists x) \cup (r_{n-1}x \cap \exists x \cap r_{n-1}\neg\exists x)) \\ &= r_{n-1}x \cup r_{n-1}(r_{n-1}x \cap \exists x \cap r_{n-1}\neg\exists x) \\ &= r_{n-1}x \cup (r_{n-1}x \cap r_{n-1}\exists x \cap r_{n-1}\neg\exists x) = r_{n-1}x, \end{aligned}$$

hence (E1).

Verification of (E2):

$$\begin{aligned} r_1\exists(x \wedge \exists y) &= r_1\exists((x \cap \exists y) \cup (r_{n-1}x \cap \exists y \cap r_{n-1}\neg\exists y)) \\ &= \exists r_1((x \cap \exists y) \cup (r_{n-1}x \cap \exists y \cap r_{n-1}\neg\exists y)) \\ &= \exists(r_1(x \cap \exists y) \cup r_1(r_{n-1}x \cap \exists y \cap r_{n-1}\neg\exists y)) \\ &= \exists(r_1(x \cap \exists y) \cup (r_{n-1}x \cap r_1\exists y \cap r_{n-1}\neg\exists y)) \\ &= \exists(r_1(x \cap \exists y) \cup (r_{n-1}x \cap r_1\exists y \cap \neg r_1\exists y)) \\ &= \exists(r_1(x \cap \exists y) \cup (r_{n-1}x \cap \mathbf{0})) \\ &= \exists r_1(x \cap \exists y) = r_1\exists(x \cap \exists y) = r_1(\exists x \cap \exists y), \end{aligned}$$

and

$$\begin{aligned} r_1(\exists x \wedge \exists y) &= r_1((\exists x \cap \exists y) \cup (r_{n-1}\exists x \cap \exists y \cap r_{n-1}\neg\exists y)) \\ &= r_1(\exists x \cap \exists y) \cup r_1(r_{n-1}\exists x \cap \exists y \cap r_{n-1}\neg\exists y) \\ &= r_1(\exists x \cap \exists y) \cup (r_{n-1}\exists x \cap r_1\exists y \cap r_{n-1}\neg\exists y) \\ &= r_1(\exists x \cap \exists y) \cup (r_{n-1}\exists x \cap r_1\exists y \cap \neg r_1\exists y) \\ &= r_1(\exists x \cap \exists y) \cup (r_{n-1}\exists x \cap \mathbf{0}) = r_1(\exists x \cap \exists y), \end{aligned}$$

by (F3), the properties of r_1, r_{n-1} and (F2), hence $r_1\exists(x \wedge \exists y) = r_1(\exists x \wedge \exists y)$; similarly, we obtain $r_{n-1}\exists(x \wedge \exists y) = r_{n-1}(\exists x \wedge \exists y)$, hence (E2).

Verification of (E3). With (F3), we obtain

$$r_1(r_1\exists x) = r_1(\exists r_1 x) \quad \text{and} \quad r_{n-1}(r_1\exists x) = r_{n-1}(\exists r_1 x)$$

and

$$r_1(r_{n-1}\exists x) = r_1(\exists r_{n-1} x) \quad \text{and} \quad r_{n-1}(r_{n-1}\exists x) = r_{n-1}(\exists r_{n-1} x),$$

hence (E3). □

Remarks. (1) We can construct a 3-L.M.a. from a monadic n-L.M.a. for each $n \geqslant 2$. If $n \geqslant 3$, this follows by applying successively Construction 1 and Construction $\mathscr{L}'$ of this paper, while for $n = 2$ we can apply directly the construction of A. Monteiro, since every (monadic) Boolean algebra is a (monadic) 2-L.M.a.

(2) If $(A'; \cup, \cap, \neg, r_1, r_2, \ldots, r_{n-1}, \mathbf{1}, \exists)$ is a monadic n-L.M.a., then $(A'; \vee, \wedge, \neg, r_{n-1}, \mathbf{1}, \exists)$ is not a monadic 3-L.M.a. (for example, (L2) is not verified), where $\vee, \wedge$ are those defined in this section.

(3) If $(A'; \exists)$ is a monadic n-L.M.a., $n \geqslant 3$, then $(B(A'), \exists)$ is a monadic Boolean algebra, where $B(A') = \{x \in A' \mid x = r_i x, i = \overline{1, n-1}\}$. By remark (1), from $(A', \exists)$ we obtain a 3-L.M.a. L and by the construction of A. Monteiro, from $(B(A'), \exists)$ we obtain a 3-L.M.a. L'. We can prove that L' is a subalgebra of L.

Construction 2

Let $(A', \exists)$ be a monadic n-L.M.a., $n > 3$. We define in the n-L.M.a. A' [3, 4] the following relations:

$$R_1, R_2, \ldots, R_{(n-1)/2}, \quad \text{if } n \text{ is odd},$$

and

$$R_1, R_2, \ldots, R_{n/2} \quad \text{if } n \text{ is even},$$

respectively, by

$$xR_1 y \Leftrightarrow (r_1 x = r_1 y \text{ and } r_{n-1} x = r_{n-1} y),$$
$$xR_2 y \Leftrightarrow (r_2 x = r_2 y \text{ and } r_{n-2} x = r_{n-2} y),$$
$$\vdots$$
$$xR_{(n-1)/2} y \Leftrightarrow (r_{(n-1)/2} x = r_{(n-1)/2} y \text{ and } r_{(n+1)/2} x = r_{(n+1)/2} y),$$
$$xR_{n/2} y \Leftrightarrow (r_{n/2} x = r_{n/2} y).$$

We denote by $A_i = A'/_{R_i} = \{\hat{x}(R_i) \mid x \in A'\}$, $i = 1, \ldots, k$, with $k = (n-1)/2$ or $k = n/2$, and we define on A_i the following operations [3, 4]:

$$\hat{x}(R_i) \vee \hat{y}(R_i) = (x \cup y)\hat{}(R_i),$$
$$\hat{x}(R_i) \wedge \hat{y}(R_i) = (x \cap y)\hat{}(R_i),$$
$$-\hat{x}(R_i) = (\neg x)\hat{}(R_i),$$
$$s_1\hat{x}(R_i) = (r_i x)\hat{}(R_i),$$
$$s_2\hat{x}(R_i) = (r_{n-i} x)\hat{}(R_i)$$

(we remark that for n even and $i = n/2$, $r_i x = r_{n-i} x = r_{n/2} x$, hence $s_1 = s_2$),

$$\hat{\mathbf{1}}(R_i) = \{x \in A' \mid xR_i\mathbf{1}\}.$$

In the monadic n-L.M.a. $(A', \exists)$ we take the above relations and, in the corresponding sets A_i, we take the above operations and one more:

$$\exists\hat{x}(R_i) = (\exists x)^\wedge(R_i).$$

Remarks. (1) $\hat{\mathbf{1}}(R_1) = \{\mathbf{1}\}$. Indeed,

$$\begin{aligned} x \in \hat{\mathbf{1}}(R_1) &\Leftrightarrow xR_1\mathbf{1} \\ &\Leftrightarrow (r_1 x = r_1\mathbf{1} = \mathbf{1} \text{ and } r_{n-1}x = r_{n-1}\mathbf{1} = \mathbf{1}) \\ &\Leftrightarrow (r_1 x = \mathbf{1} = r_{n-1}x) \Leftrightarrow x = \mathbf{1}, \end{aligned}$$

by $\mathbf{1} = r_1 x \leqslant x \leqslant r_{n-1} x = \mathbf{1}$.

(2) $\hat{\mathbf{0}}(R_1) = \{\mathbf{0}\}$.

(3) If n is even, then $\hat{x}(R_{n/2}) = (r_{n/2}x)^\wedge(R_{n/2})$, since $r_{n/2}(r_{n/2}x) = r_{n/2}x$, hence $r_{n/2}xR_{n/2}x$.

(4) It follows that in $A_{n/2}$ (n is even) we have $s_2\hat{x}(R_{n/2}) = s_1\hat{x}(R_{n/2}) = (r_{n/2}x)^\wedge(R_{n/2}) = \hat{x}(R_{n/2})$.

Theorem 3. *With the above conditions, we have*

(a′) *The system* $(A_i = A'/_{R_i}; \vee, \wedge, -, s_2, \hat{\mathbf{1}}(R_i))$ *is, for every* $i = 1, \ldots, k$, $k = (n-1)/2$, *if* n *is odd and* $k = n/2$, *if* n *is even, a* 3*-L.M.a.*

(b′) *The pair* $(A_i, \exists)$ *is, for every* $i = 1, \ldots, k$, *a monadic* 3*-L.M.a.*

Proof. (a′) is proved in [3, 4] and (b′) is immediate. □

Corollary 1. *If* n *is even,* $A_{n/2}$ *is a Boolean algebra* (*see the above Remark* (4)).

Corollary 2 ([3, 4]). *The* 3*-L.M.a.* $(A_1 = A'/_{R_1}; \vee, \wedge, -, s_2, \hat{\mathbf{1}}(R_1) = \{\mathbf{1}\})$ *coincides with the* 3*-L.M.a.* $(A = A'/_R; \vee, \wedge, -, s_2, 1 = \{\mathbf{1}\})$, *as structure.*

Corollary 1′. *If* n *is even* (*see Corollary* 1) $(A_{n/2}, \exists)$ *is a monadic Boolean algebra.*

Corollary 2′. *The monadic* 3*-L.M.algebras,* $(A_1, \exists)$ *and* $(A, \exists)$ *have the same structure* (*see Corollary* 2).

Let us consider now, for n odd, the cartesian product $P = A_1 \times A_2 \times \cdots \times A_{(n-1)/2}$ of the above monadic 3-L.M.algebras, with the induced operations $\vee_P, \wedge_P, -_P, s_2^P, 1^P, \exists^P$. It is easy to prove that, for $n > 3$, $(P; \vee_P, \wedge_P, -_P, s_2^P, 1^P)$ is a 3-L.M.a. [3, 4], and $(P, \exists^P)$ is a monadic 3-L.M.a.

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Note added in proof

The author has generalized the $\mathscr{L}'$ construction on monadic $(1+\theta)$-valued Lukasiewicz–Moisil algebras.

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